The Thermal Recoil Force

Some Background Notes

Why is there a recoil force?

- **Particle picture**
  Photons carry momentum:
  $|\mathbf{p}| = h\nu / c$.
  The momentum vector points in the direction of the photon's path.
- **Wave picture**
  Energy flow of the EM field is described by the Poynting-vector $\mathbf{S}$; momentum is $\mathbf{p} = \mathbf{S} / c$.

The Language of (Classical) EM

- Electromagnetic stress-energy tensor:

$$T^{\mu\nu} = \begin{pmatrix} u & \mathbf{S} \\ \mathbf{S} & \mathbf{T} \end{pmatrix}.$$

- Conservation of energy-momentum:

$$\nabla_\mu T^{\mu\nu} = 0.$$

(NB: From now on, we use $c = 1$.)

The Language of EM (cont'd)

- u is the energy density of the EM field:

  $$u = \mathbf{E}^2 + \mathbf{H}^2.$$

- $\mathbf{S}$ is the Poynting vector:

  $$\mathbf{S} = \mathbf{E} \times \mathbf{H}.$$

- $\mathbf{T}$ is the Maxwell stress tensor:

  $$\mathbf{T} = \mathbf{EE} + \mathbf{HH} - \tfrac{1}{2}(\mathbf{E}^2 + \mathbf{H}^2)\mathbf{I}.$$

  ($\mathbf{I}$ is the identity tensor; $\mathbf{xy}$ is the dyadic product of two vectors.)

  For "rays of light" (plane waves, spherical waves)

  $$\mathbf{T} = \mathbf{SS} / |\mathbf{S}|,$$

  for each individual "ray".

The Language of EM (cont'd)

- Energy conservation:

$$\partial u / \partial t - \nabla \cdot \mathbf{S} = 0,$$

- Momentum conservation:

$$\partial \mathbf{S} / \partial t - \nabla \cdot \mathbf{T} = 0.$$

The Language of Heat

- The (integrated) intensity I is the flow of energy across surface element $\mathbf{dA}$, in a time dt, in the solid angle $d\omega$ around direction $\mathbf{n}$:

  $$dE = I(\mathbf{x}, t, \mathbf{n})\mathbf{n} \cdot \mathbf{dA}\ d\omega\ dt.$$

  We can write

  $$\mathbf{S}(\mathbf{x}, t) = \int I(\mathbf{x}, t, \mathbf{n})\mathbf{n}\ d\omega,$$

  and, after applying Gauss's theorem, get

  $$dE/dt = \int \mathbf{S}(\mathbf{x},t) \cdot \mathbf{dA} = \int \nabla \cdot \mathbf{S}(\mathbf{x},t)\ dV,$$

  as expected.

The Language of Heat (cont'd)

- We can introduce the quantity

  $$q(\mathbf{x}, t) = \mathbf{S}(\mathbf{x}, t) \cdot \mathbf{a} = \int I(\mathbf{x}, t, \mathbf{n}) \cdot \mathbf{a}\, d\omega,$$

  where $\mathbf{a}$ is the unit normal to $\mathbf{dA}$ (i.e., $\mathbf{dA} = \mathbf{a}dA$). Thereafter, the energy flow across a surface can be written as

  $$Q = dE / dt = \int q(\mathbf{x}, t)\, dA.$$

- At the surface of a Lambertian emitter, I is independent of $\mathbf{n}$ and can be taken outside the integral sign:

  $$q(\mathbf{x}, t) = I(\mathbf{x}, t) \int \mathbf{a}\, d\omega = \pi I(\mathbf{x}, t).$$

The Language of Heat (cont'd)

- The radiation pressure tensor is defined as

$$\mathbf{P}(\mathbf{x}, t) = \int I(\mathbf{x}, t, \mathbf{n})\mathbf{nn}\, d\omega.$$

  Note form similar to the Maxwell stress tensor.

- By momentum conservation, the change in momentum (i.e., force) can be written as

$$\mathbf{F} = \mathbf{dp}/dt = \int \nabla \cdot \mathbf{P}(\mathbf{x},t)\, dV = \oint_{\partial V} \mathbf{P}(\mathbf{x},t) \cdot \mathbf{dA}.$$

The Language of Heat (cont'd)

- For a Lambertian emitter, I is independent of $\mathbf{n}$ and can be taken outside the integral sign:

$$\mathbf{P}(\mathbf{x}, t) = I(\mathbf{x}, t) \int \mathbf{nn}\, d\omega.$$

Thereafter,

$$\mathbf{F} = \oint_{\partial V} I(\mathbf{x},t) \int \mathbf{nn} \cdot \mathbf{a}\ d\omega\, dA = \frac{2}{3} \oint_{\partial V} q(\mathbf{x},t) \mathbf{dA}.$$

- Restoring c, the recoil force $\mathbf{f}$ acting on a Lambertian surface element at $\mathbf{x}$ with normal $\mathbf{a}$ is

$$\mathbf{f} = -(2/3c)\, q(\mathbf{x}, t)\mathbf{a}.$$

The Language of Heat (cont'd)

- We can even calculate torque:

$$\boldsymbol{\tau} = -\frac{2}{3c}\oint_{\partial V} q(\mathbf{x},t)(\mathbf{x}-\mathbf{x}_0)\times\mathbf{dA}.$$

where $\mathbf{x}_0$ is the emitter's center-of-gravity.

Sources of Heat

- Spacecraft like Pioneer have internal heat sources. Far from the Sun, external heating is negligible.
- Heat convection is negligible; there are no coolants, no significant fuel flow.
- Two forms of internal heat transport from sources to external (radiating) surfaces remain:
  - Conduction,
  - Radiation.

Conduction

- Conduction is governed by Fourier's equation:

$$\mathbf{q} = -k\nabla T.$$

- Further,

$$\nabla \cdot \mathbf{q} = b - C_h \rho \partial T / \partial t.$$

where b is the volumetric heat release.

- The heat conducted to a surface must equal the heat radiated by that surface:

$$q = \mathbf{q} \cdot \mathbf{a}.$$

Heat Sources

- For compact heat sources of power $B_i(t)$ located at $\mathbf{x}_i$,

$$b = \Sigma B_i(t)\delta^3(\mathbf{x} - \mathbf{x}_i).$$

- Steady state: $\partial T / \partial t = 0$. In this case, we have

$$\int b\, dV = \sum B_i = \int \nabla \cdot \mathbf{q}\, dV = \int \mathbf{q} \cdot \mathbf{dA} = \int q\, dA.$$

as we would expect from energy conservation.

Radiation

- Radiative exchange can be calculated by integrating across all solid angles between two surfaces:

$$Q_{1\to 2} = \int (q_1 - q_2) r^{-2} \cos\theta_1 \cos\theta_2 \, dA_1 \, dA_2.$$

- But q on the surface can be calculated using the Stefan-Boltzmann law:

$$q = \sigma\varepsilon T^4.$$

- Boundary condition: emitter is surrounded by an infinitely large blackbody sphere at T_{CMB}.

Computing the Recoil Force

- Two ways of practical calculation
  - We can compute the recoil force at the emitting surfaces (taking into account interaction between external surfaces of a nonconvex emitter); or
  - We can compute the momentum transferred to an infinitely large (in practice: very large) “control volume”.

What determines the force?

- It's not the total amount of heat that matters; it is the *difference* (anisotropy) in heat radiated in different directions.
- If heat sources are known, the total amount can be computed easily (energy conservation.)
- The difference is much harder to calculate, and may be sensitive to small errors.
- Nevertheless, energy conservation can be used to calibrate results; and
- In some cases, a linear model may work.

Power vs. Temperature

- Note the dominant use of “power language” as opposed to “temperature language” in preceding slides.
- We’re not interested in maintaining operating temperatures (the typical task for thermal engineering).
- Force is a function of power emitted, only indirectly (through a fourth-power relationship) a function of temperature.

In terms of numbers...

- Total amount of heat radiated by Pioneer is in excess of 2 kW even at the end of mission;
- Amount needed to produce constant acceleration of ~8.74 m/s^2 is just ~65 W.
- This is an anisotropy of only ~3%!
- A small error (say, 5%) in the estimation of total radiated heat translates into an unacceptable error (~ 167%) in the estimated recoil force.
- Not insurmountable, but definitely a difficulty.