

On the Unification Problem in Physics*

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In the general theory of relativity, in order to characterize world events, the fundamental metric tensor $g_{\mu\nu}$ of the 4-dimensional world manifold, interpreted as the the tensor potential of gravitation must be introduced separately from the electromagnetic four-potential q_μ .

The dualistic nature of gravitation and electricity still remaining here does not actually destroy the ensnaring beauty of either theory but rather affords a new challenge towards their triumph through an entirely unified picture of the world.

A few years ago H. WEYL¹ put forward a surprisingly courageous attack to the solution of these problems which belong among the most magnificent ideas of the human spirit. Through the radical reexamination of the geometric grounds he obtained, in addition to the tensor $g_{\mu\nu}$, a kind of fundamental metric vector and interpreted it as the electromagnetic potential q_μ : There the complete world metric is claimed to be the common source of all the natural phenomena.

Here we strive for the same goal in a different way.

If we disregard the difficulty which is associated with the practical application of H. WEYL's profound theory, it is thought to be quite idealistically possible to realize completely the unification idea in which the gravitational and electromagnetic fields stem from a single universal tensor. — I would now like to show that such an intimate combination of the two forces in the world appears to be possible in principle.

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The rotational form of the electromagnetic field components $F_{\nu\lambda}$ and also the apparent formal equivalence in the structure of the gravitational and electromagnetic equations² lead us to the conjecture that $\frac{1}{2}F_{\nu\lambda} =$

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¹Sitzungsber. d. Berl. Acad. 1918 p.465.

²In this respect refer to H. THIRRING. Phys. Ztschr. 19 p.204.

$\frac{1}{2}(q_{\varkappa,\lambda} - q_{\lambda,\varkappa})^1$ may somehow be equal to the truncated three-index quantities $\begin{bmatrix} i\lambda \\ \varkappa \end{bmatrix} = \frac{1}{2}(g_{i\varkappa,\lambda} + g_{\varkappa\lambda,i} - g_{i\lambda,\varkappa})$. Provided there is a room for this idea, one may be drawn with greater confidence to a path that previously seemed less attractive: Since in a four-dimensional world, beyond the three-index quantities used as the field components of gravitation, no further such quantities exist, such an interpretation of $F_{\varkappa\lambda}$ is hardly supported unless one makes the otherwise extremely odd decision to ask for help from a new fifth dimension of the world.

Although our rich physical experience obtained so far provides little suggestion to such an extra world-parameter, we are certainly free to consider our space-time to be a four-dimensional part of R_5 ; one then has to take into account the fact that we are only aware of the space-time variability of state parameters, by making their derivatives with respect to the new parameter vanish or by considering them to be small as they are of higher order (“cylinder condition”). Misgivings about the retrogressive introduction of the fifth dimension are rendered groundless by connecting the world parameter to the three-index quantities.

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We therefore enter into R_5 and extend EINSTEIN’s ansatz to R_5 : at the same time, we introduce the new parameter x^0 in addition to the usual x^1 to x^4 . If the fundamental metric tensor of this R_5 is expressed as g_{rs}^2 , the three-index quantities $\begin{bmatrix} ik \\ l \end{bmatrix}$, denoted here by $-\Gamma_{ikl}$, become, by virtue of the cylinder condition,

$$\begin{aligned} 2\Gamma_{\varkappa\lambda\mu} &= g_{\varkappa\lambda,\mu} - g_{\lambda\mu,\varkappa} - g_{\mu\varkappa,\lambda} && \text{(as before),} \\ 2\Gamma_{0\varkappa\lambda} &= g_{0\varkappa,\lambda} - g_{0\lambda,\varkappa}, && 2\Gamma_{\varkappa\lambda 0} = -(g_{0\varkappa,\lambda} + g_{0\lambda,\varkappa}), && (1) \\ 2\Gamma_{00\varkappa} &= g_{00,\varkappa}, && 2\Gamma_{0\varkappa 0} = -g_{00,\varkappa}, && 2\Gamma_{000} = 0. \end{aligned}$$

This result is at first sight hardly encouraging: Indeed $\Gamma_{0\varkappa\lambda}$ appear in the form of a rotation, but the ten $\Gamma_{\varkappa\lambda 0}$, which would have to be of electric nature according to our interpretation, are in danger of being in the way. Nevertheless, we further investigate the above result and, in order to keep

¹By indices divided by a comma is meant the differentiation with respect to the corresponding world-parameter.

²Latin indices always run from 0 to 4 and Greek ones only from 1 to 4.

$\Gamma_{0\alpha\lambda}$ proportional to $F_{\alpha\lambda}$ we set

$$g_{0\alpha} = 2\alpha q_{\alpha}, \quad g_{00} = 2\mathbf{g}, \tag{2}$$

so that the fundamental metric tensor of R_5 substantially becomes the gravitational tensor potential framed by the electromagnetic four-potential: The role of the component \mathbf{g} in the corner remains undetermined for the time being. Introducing the shorthand $\Sigma_{\alpha\lambda}$ for the sum $q_{\alpha,\lambda} + q_{\lambda,\alpha}$ corresponding to $F_{\alpha\lambda}$, we have

$$\Gamma_{0\alpha\lambda} = \alpha F_{\alpha\lambda}, \quad \Gamma_{\alpha\lambda 0} = -\alpha \Sigma_{\alpha\lambda}, \quad \Gamma_{00\alpha} = -\Gamma_{0\alpha 0} = \mathbf{g}_{,\alpha}. \tag{3}$$

Consequently the electromagnetic field $F_{\alpha\lambda}$, its “associated” field $\Sigma_{\alpha\lambda}$ and the gradient¹ of \mathbf{g} use up the thirty-five new three-index symbols (five of which vanish). Moreover, from the comprehensive equality

$$(\Gamma_{ikl} + \Gamma_{kli} + \Gamma_{lik})_{,m} = \Gamma_{mik,l} + \Gamma_{mkl,i} + \Gamma_{mli,k} \tag{4}$$

arise the well-known relations by virtue of the cylinder condition:

$$F_{\alpha\lambda,\mu} + F_{\lambda\mu,\alpha} + F_{\mu\alpha,\lambda} = 0 \quad \text{and} \quad \mathbf{g}_{,\alpha\lambda} = \mathbf{g}_{,\lambda\alpha}. \tag{4a}$$

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We now restrict, as usual, the choice of parameters by $g = |g_{rs}| = -1$ and let g_{rs} differ only a little bit from the “Euclidean” value $-\delta_{rs}$ (approximation I). With $\Gamma_{ik}^l = -\begin{Bmatrix} ik \\ l \end{Bmatrix} = -\Gamma_{ikl}$ the interesting components of the two four-index tensors:

$$\begin{aligned} \{\alpha\lambda, \mu 0\} &= \alpha F_{\alpha,\mu}^{\lambda}, & \{\alpha 0, 0\lambda\} &= -\mathbf{g}_{,\alpha\lambda}, \\ \{\alpha\lambda, 00\} &= \{\alpha 0, 00\} = \{00, 00\} = 0. \end{aligned} \tag{5}$$

Fortunately the associated field in Eq. (3) does not participate here: Among electric quantities only derivatives of the fields appear which determine the curvature of R_5 . Further, by making the contracted tensor $R_{ik} = \{ir, rk\}$, we find, according to our assumptions (in the well-known notation):

$$\begin{aligned} R_{\mu\nu} &= \Gamma_{\mu\nu,\rho}^{\rho}, & (\text{as earlier}), \\ R_{0\mu} &= -\alpha \Delta v_{\mu} F, \\ R_{00} &= -\square \mathbf{g}. \end{aligned} \tag{6}$$

Therefore the fifteen components of the curvature tensor on the left-hand side reduce into: 1. the ordinary field equations of gravitation, 2. the basic

¹In the four-dimensional sense.

electromagnetic equations and 3. a POISSON equation for the yet unexplained g . Therein lies the foremost justification of our ansatz and the hope to consider gravitation and electricity as manifestations of a universal field.

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For the energy-momentum tensor of matter, dominating the right-hand side of the field equations in R_5 , holds the following under approximation I:

$$T_{ik} = T^{ik} = \mu_0 u^i u^k, \tag{7}$$

$$(\mu_0 = \text{rest mass density, } u^r = \frac{dx^r}{ds}, \quad ds^2 = g_{lm} dx^l dx^m);$$

Since now (for all three types of field equations) $R_{0\mu} = -\varkappa T_{0\mu}$, it follows from the MAXWELL equations that according to Eq. (6) for the components of the four-current:

$$\mathbf{I}^\mu = \rho_0 v^\mu = \frac{\varkappa}{\alpha} T_{0\mu} = \frac{\varkappa}{\alpha} \mu_0 u^0 u^\mu \tag{8}$$

$$(\rho_0 = \text{rest charge density, } v^\rho = \frac{dx^\rho}{d\sigma}, \quad d\sigma^2 = g_{\lambda\mu} dx^\lambda dx^\mu);$$

the space-time energy-momentum tensor is thus framed essentially by the current density.

We then continue further investigation under the assumption $u^0, u^1, u^2, u^3 \ll 1, u^4 \sim 1$ (approximation II). This implies not only a small velocity but also a very tiny specific charge $\frac{\rho_0}{\mu_0}$ of moving matter; in fact, as thereupon

$d\sigma^2 \sim ds^2, v^\rho \sim u^\rho$, it follows from Eq. (8) if one sets¹ $\alpha = \sqrt{\frac{\varkappa}{2}} = 3.06 \times 10^{-14}$:

$$\rho_0 = \frac{\varkappa}{\alpha} \mu_0 u^0 = 2\alpha \mu_0 u^0 \ll \mu_0. \tag{8a}$$

First of all this equation teaches us again that in this case we need to understand the electric charge essentially as the fifth component of the energy-momentum of the matter “moving across” the space $x^0 = \text{const.}$: A further merger of two formerly heterogeneous basic concepts thus appears.

Because finally $T_{00}, T_{11}, T_{22}, T_{33} \sim 0$ in approximation II, we find according to Eq. (7):

$$T = g^{ik} T_{ik} = -T_{44} = -\mu_0, \tag{9}$$

¹According to the equation of motion; see the next section.

so that for the ordinary form of the field equation of the first kind:

$$R_{00} = -R_{44} = \frac{\varkappa}{2}\mu_0. \tag{10}$$

The corner potential \mathfrak{g} is also proven, due to Eq. (6), to be essentially minus the gravitational potential, while $\mathfrak{G} = \frac{g_{44}}{2}$ keeps the former meaning.

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After we thus disposed of the standard quantities in the field equations in a satisfactory way, we still face with the question whether the “geodesic” equation of motion in R_5 ,

$$\dot{u}^l = \frac{du^l}{ds} = \Gamma_{rs}^l u^r u^s \tag{11}$$

then represents the motion of charged matter in the gravitational and electromagnetic field in accord with experiments. In approximation II this is immediately the case: Due to the interchangeability of ds and $d\sigma$ one obtains according to Eq. (3):

$$\bar{v}^\lambda = \frac{dv^\lambda}{d\sigma} = \Gamma_{\rho\sigma}^\lambda v^\rho v^\sigma + 2\alpha F_{\varkappa}^\lambda u^0 u^\varkappa - \mathfrak{g}_{,\lambda} u^{02}, \tag{11a}$$

i.e., due to the smallness of the term with u^{02} for the force density one finds

$$\pi^\lambda = \mu_0 \bar{v}^\lambda = \Gamma_{\rho\sigma}^\lambda T^{\rho\sigma} + F_{\varkappa}^\lambda \mathbf{I}^\varkappa \quad \left(\alpha = \sqrt{\frac{\varkappa}{2}} \text{ adapted; see footn.} \right). \tag{12}$$

The total force thus splits automatically into a gravitational and electromagnetic part of the ordinary form.

Finally, for the 0-component of Eq. (11) there remains only

$$\dot{u}^0 = \alpha \Sigma_{44} = 2\alpha q_{4,4}, \tag{11b}$$

so that, in approximation II, the quasi-static $\frac{d}{dx^4} \left(\frac{\rho_0}{\mu_0} \right) = 2\varkappa q_{4,4}$ ¹ becomes small in higher orders: The necessary constancy of ρ_0 seems to be guaranteed accordingly.

Hence also for the equations of motion the associated field remains insignificant in our approximation.

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¹Cf. (8a).

If approximation II was in accord with the reality, the unification theory under investigation would be executed, as mentioned earlier, substantially in a satisfactory manner: A unique potential tensor generates a universal field which splits, in ordinary condition, into a gravitational and electromagnetic part.

Now, however, the matter at least in its smallest building blocks is not at all weakly charged: following WEYL's expression, its "macroscopic rest" is confronted with its "microscopic restlessness", and this holds, in the above-mentioned view, quite specifically for the new world-parameter x^0 : For the electron or H-nucleus, $\frac{\rho_0}{\mu_0}$ is not small and so the "velocity" component u^0 is not at all small! In the form constrained by approximation II the theory can at best coarsely describe macroscopic phenomena, and a fundamental problem arises concerning its very applicability to those elementary particles.

If one now, nevertheless, tries to describe the motion of the electron by geodesics in R_5 , one immediately meets with the serious difficulty¹ which threatens to destroy the foundation laid here. It lies simply in the fact that, with the rigid application of the previous assumptions, u^0 for the electrons becomes enormously large owing to $\frac{e}{m} = 1.77 \times 10^7$ (reduced in light seconds) so that the last term in Eq. (IIa), instead of vanishing, assumes a value that exceeds everything and defies all experience, provided that everything remains formally as before. Now the larger value of u^0 indeed implies modifications of the theory anyhow (the interchangeability of ds and $d\sigma$ is lost), but it seems almost impossible to develop the theory only in the old framework without any new hypotheses.

Nevertheless – with full reservations – I believe in finding, clearly in the following direction, a way which provides a fully satisfactory point of view leading to the goal. Since $R_{00} \sim -R_{44}$ even for arbitrary u^0 provided that the velocity of the matter which generates fields is not too large, two gravitational terms in Eq. (IIa) take opposite signs if the nature of x^0 which has been completely irrelevant is determined suitably. And it then seems that, apart from the gravitational constant \varkappa which is slightly questionable anyhow, a reconciliation is possible between two compelling magnitudes for which gravitation is obtained as a kind of the difference effect. One is tempted to be bribed by this possibility with the prospect that the role of a statistical quantity may be assigned to that constant. At the moment the consequence of this hypotheses is not yet examined sufficiently; other

¹I wish to thank Mr. Einstein for his valuable interest in the origin of the above assumptions and the suggestion of the inconsistencies described here.

possibilities should also be searched for. After all, what threatens all the ansatz which demand universal validity is the sphinx of modern physics – quantum theory.

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In spite of full recognition of the aforementioned physical as well as epistemological difficulties piled up in front of our understanding developing here, it is rather difficult for us to believe that, in all those relations which in their formal unity seem hard to be replaced by anything else, only a capricious accident performs its alluring play. If it could be proven some day that there exists more behind the presumed relations than merely meaningless formalism, then this would certainly imply a new triumph for EINSTEIN's general theory of relativity, whose appropriate application to the five-dimensional world comes into question.